Minkowski sum of $\mathcal{HV}$-polytopes in $\mathbb{R}^n$

4th Annual International Conference on Computational Mathematics, Computational Geometry & Statistics

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Singapore, January 26th, 2015
Modelling the manufacturing defects

- CAD model with perfect surfaces VS Tolerance Analysis model

- Objective: control the variation of $d$ by taking into account the allowable variations of all parts $d_{min} \leq d \leq d_{max}$?
Modelling the manufacturing defects

Modelling the fabricated surface (FS)

- FS is the CAD surface translated and rotated in $\mathbb{R}^6$
- FS admissible locations to remain in the tolerance zone is a polytope

Computing the cumulative stack-up of variations implies calculating the Minkowski sums of polytopes.

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Minkowski sum of $\mathcal{HV}$-polytopes in $\mathbb{R}^n$
What are polytopes?

Minkowski-Weyl theorem states that both definitions are equivalent.

**Definition of the $\mathcal{V}$-description**

$P$ is the convex combination of a finite set of points

\[ P = \left\{ x \in \mathbb{R}^n, x = \sum_{i=1}^{k} \alpha_i a_i \right\} \]

$a_i \in \mathbb{R}^n, \alpha_i \in \mathbb{R}^+, \sum_{i=1}^{k} \alpha_i = 1$

**Definition of the $\mathcal{H}$-description**

$P$ is the bounded intersection of a finite set of half-spaces

\[ P = \bigcap_{u=1}^{l} \bar{H}^+_u \]
Computing the Minkowski sum of two polytopes

- $A$ and $B$, two $\mathbb{R}^n$-polytopes of full dimension $n$
- $C = A + B = \{c = a + b, a \in A, b \in B\}$

Previous works

- Fukuda in 2004 described an algorithm to sum 2 skeletons ($V$-description+edges)
- Delos & Teissandier in 2014 wrote an algorithm to sum to $V$-polytopes
- Matlab and qhull provide interfaces to remove the interior of a cloud of points
Computing the Minkowski sum of two $\mathcal{HV}$-polytopes

The algorithm is based on the normal fans refinement

- It intersects dual cones from $A$ and $B$
- It needs both polytopes $\mathcal{HV}$-description
Summing a cube with an octaedron in $\mathbb{R}^3$

1. Polytopes
2. Normal fans & dual cones
3. Common refinement
4. Sum

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Minkowski sum of $\mathcal{HV}$-polytopes in $\mathbb{R}^n$
Algorithm 1 Calculate $C = A + B$ in $\mathbb{R}^n$

Require: List of dual cones of $A$ \{ $C_D(a_i), a_i \in \mathcal{V}_A$\}

Require: List of dual cones of $B$ \{ $C_D(b_j), b_j \in \mathcal{V}_B$\}

for all $a_i \in \mathcal{V}_A$ and $b_j \in \mathcal{V}_B$ do
  Compute $C_D(a_i, b_j) = C_D(a_i) \cap C_D(b_j)$
  // Check if the intersection is of full dimension
  if $\dim(C_D(a_i, b_j)) = n$ then
    $c_{ij} = (a_i + b_j)$ is a vertex of $C$
    Get the half-spaces passing through $c_{ij}$ from $C_D(a_i, b_j)$ edges
  end if
end for
The algorithm has been implemented in a software named *politopix*

<table>
<thead>
<tr>
<th>P1 + P2</th>
<th>P3</th>
<th>time</th>
</tr>
</thead>
<tbody>
<tr>
<td>(40,224) + (16,144)</td>
<td>(7732,9572)</td>
<td>53s</td>
</tr>
<tr>
<td>(16,194) + (18,256)</td>
<td>(7998,15546)</td>
<td>119s</td>
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<tr>
<td>(215,1320) + (14,96)</td>
<td>(3954,10248)</td>
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<tr>
<td>(1584,2048) + (1584,2048)</td>
<td>(1736,53836)</td>
<td>114min</td>
</tr>
</tbody>
</table>

- Computing P1 + P2 = P3 in $\mathbb{R}^6$, with the number of half-spaces and vertices between brackets
- 64 bits OS installed on a PC Intel(R) Xeon(R) CPU E3-1270 V2 @ 3.50 GHz with 32,0 Go RAM
Computing the Minkowski sum of two $HV$-polytopes

To optimize the algorithm we are aware of 2 things:

- Most of the dual cones intersections return the origin:
  \[ C_D(a_i) \cap C_D(b_j) = \{ O \} \]
- We can use neighborhood properties of the dual cones to skip useless cone / cone intersections
- We develop the notion of polyhedrical cap
Conclusion

- We presented an algorithm summing HV-polytopes
- It is frequently used in the field of tolerance analysis on real life cases
- It’s been developed in C++ under the Gnu General Public License v3.0
- You can download it at http://i2m.u-bordeaux.fr/politopix